Errata and notes for vol 1 and vol 2

3rd March 2023

Volume One: Arithmetic Equivalents

• page xx about line -8: add to the acknowledgments Jim Jacobson, Roger MacKay, Yuri Matiyasevich, Terence Tao, Ken Ono, Nilo de Roock and Xiaolong Wu.

• page 2 line -3: Euler's form, view and application of the functional equation is clearly described in Section 5.2 of V. S. Varadarajan's *Euler through time: a new look at old themes*, AMS, 2006. Indeed, he did not restict s to integer values but proved the functional equation was true for integers, for 1/2, for 3/2, and assumed it to be true for all real s.

• page 16 line 19: replace $\zeta(s)$ by $(1-2^{1-s})\zeta(s)$.

• page 40 line -4: sum should be over $p \le x$ not $n \le x$.

• page 70 line -10: replace g_2 by γ_2.

• page 71 line 8: the sum should be from 2 to $n-1$.

• page 73 line 1 of Step (7): replace If$< \delta < 1-1/x$ by If $0 < \delta < 1-1/x$.

• page 74 line 9 of Step (8): include the missing term $+h\log(2\pi)$.

• page 74 line 5 of Step (9): replace $\int_0^\delta (1+y)$ by $\int_0^\delta (1+y)\ dy$.

• page 76 in formula (4.1) replace $\alpha_2 := \frac{\log^2 \xi}{\pi\sqrt{\xi}} < 0.0126$ by $\alpha_2 := \frac{\log \xi}{\pi\sqrt{\xi}} < 0.0126$.

• page 81 Theorem 4.1 statement: replace the given equation by

$$\psi(x) - x = \Omega_{\pm}(\sqrt{x})$$

See Proposition 5.2.2 in G. J. O. Jameson, *The prime number theorem*, Cambridge, 2003.

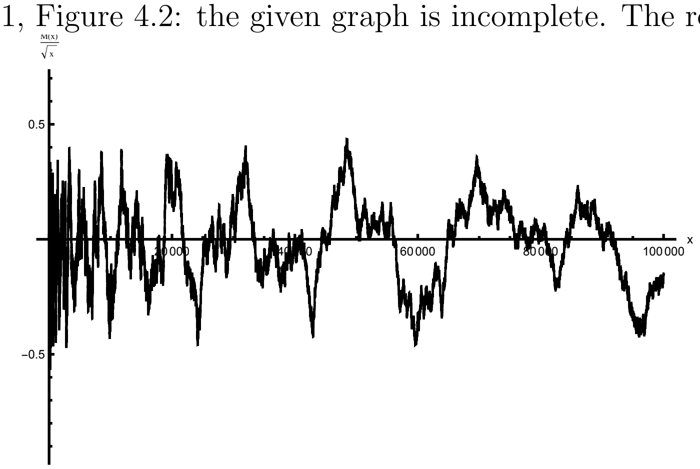
- page 85 line 14: replace the expression by

$$= -\sqrt{x}\sum_{\gamma>0}\frac{\sin(\gamma\delta)}{\gamma\delta}\frac{x^{i\gamma}-x^{-i\gamma}}{i\gamma}+O(\sqrt{x})$$

- page 87 line 12: the second formula should be replaced by

$$\pi(y_n) < \mathrm{Li}(y_n) - \frac{\sqrt{y_n}\,\mathrm{logloglog}\,y_n}{3\log y_n}\pi(y_n).$$

- page 91, Figure 4.2: the given graph is incomplete. The replacement:



- page 130 line -2: replace the expression given for $\zeta(s)$ by

$$\zeta(s) = \prod_p (1-1/p^s)^{-1}$$

- page 134 line 14: replace $a = S(x)$ by $a = \theta(x)$
- page 153 line -10: insert after $\frac{\log\left(\frac{13}{12}\right)}{\log(3)}$ the ratio $\frac{\log\left(\frac{8}{7}\right)}{\log(7)}$
- page 153 line -8: insert after $0.07285\ldots$ the number $0.06862\ldots$
- page 154 Fig. 6.2 caption: replace $F(3,\alpha)$ by $F(x,3)$ and $F(2,\alpha)$ by $F(x,2)$

- page 156 line -15: replace ϵ by ϵ_i

- page 158 Table 6.1 line 7: replace $\log_2(31/30)$ by $\log_7(8/7)$, 0.04730 by 0.06862, and 2.4612 by 2.17003

- page 158 Table 6.1 line 8: replace $\log_2(63/62)$ by $\log_2(31/30)$, 0.02308 by 0.04730, and 3.09998 by 2.46012

- page 160 line -13: replace $M < N$ by $2 < M < N$.

- page 161 Equation (6.5): replace $x^{1/k}$ by x^k.

- page 167 line 11: replace $1 < n$ by $1 \leq n$.

- page 173 lines -4 and -3: replace $-3\sqrt{2}$ by $-1.986\sqrt{2}$
- page 180 formula (7.54): replace 1.5626 by 1.31
- page 221 figure 9.2: in the uppermost box the word "Lem" should be replaced by "Thm 7.15".

- page 241 line -13 to -9: Here are paraphrased comments of Prof Y. Matiyasevich: "The proposition RH is only semidecidable, that it is, if it false, then this fact can be established by computer. The situation has been described correctly on page 2 in Volume 2.

Recently algorithms which achieve this have exhibited explicitly. For example, Adam Yedidia and Scott Aaoronson constructed a Turing machine that never stops if and only if RH is true. This machine has 5,372 states. It is just an incarnation of Shapiro criterion (Theorem 10.1 from Volume 1). See

Article: A relatively small Turing machine ....

However, at the time I was writing my book about Hilbert's tenth problem

Book: Hilbert's Tenth Problem,

I was able to use Schoenfeld's estimate on $\psi(n)$, which simplified construction of corresponding Diophantine equation. This estimate allows one to construct a corresponding Turing machine with just 744 states. See

Code: a small Turing machine.

See also

Article: By C. S. Claude and E. Claude in Complex Systems.

In addition, see the article by Brandon Fodden, *Diophantine equations and the generalized Riemann hypothesis*, Journal of Number Theory **131** (2011), 1672 1690."

• page 266 line 2 under table 10.1: replace “imaginary” by “non-trivial”.

• page 283 line -7: add “See also on the physics side the recent work of Bender, Brody and Müler, *Hamiltonian for the zeros of the Riemann zeta function*, Phys. Rev. Lett. **118** (2017), 130201, 5 pp. and the related comments of Bellissard in arXiv:1704.02644v1 [quant-ph] 9 April 2017, and the Mathematics Stack Exchange blog post entitled *Riemann hypothesis: is Bender-Brody-Müller Hamiltonian a new line of attack?*:

Link to the Stack Exchange article.

• page 290 line 4: insert “The number $n_1 = 1$ is often left off such lists.”

Volume Two: Analytic Equivalents

• page xviii about line -3: add Jim Jacobson, Roger MacKay, Yuri Matiyasevich, Terence Tao, Ken Ono.

• page 92 after line 9: Brad Rogers and Terence Tao have now shown Newman's conjecture is true, and so RH is equivalent to $\Lambda = 0$, whatever definition of Λ is used. A preprint is “The De Bruijn-Newman constant is non-negative”, arXiv:1801.05914v2[math.NT] (14 Feb 2018). A report on this work would be included in a second edition.

• page 79 theorem 5.13: it has been noted that this theorem may have a converse and thus provide an nice equivalence for RH. However a proof of the converse has not so far being found, but could exist somewhere in the literature.

• page 140 line 12: insert $\sigma \neq \frac{1}{2}$ after $0 < \sigma < 1$ and replace “or” by “for”.

• page 141: replace Step (3) by the following: Step (3) Now let RH be true and $0 < \sigma < 1$ with $\sigma \neq 0.5$. Let $\varphi(x)$ be bounded, $|\varphi(x)| \leq M$ for all $x \in \mathbb{R}$, and measurable on $\mathbb{R}$. Let $\epsilon > 0$ and $n \in \mathbb{N}$ be given. Let ψ_n be the characteristic function of $[-n, n]$ so that $\varphi \cdot \psi_n \in L_1(\mathbb{R})$. Assume for all $x \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} K_\sigma(x-y)\varphi(y)\, dy = 0.$$

Replacing $\varphi(y)$ by $\varphi(-y)$ we get $\int_{-\infty}^{\infty} K_\sigma(y+x)\varphi(y)\,dy = 0$ also. Since $K_\sigma(y)$ is real this same identity applies to the real and imaginary parts of φ, so from this point we can assume $\varphi(y)$ is also real.

Since $|\varphi(y)| \leq M$, by Theorem 8.11 the translates of $K_\sigma(y)$ are dense in $L_1(\mathbb{R})$. Therefore there exist constants $a_1, \ldots, a_J$ and $\lambda_1, \ldots, \lambda_J$ such that

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \left(\psi_n(y)\varphi(y)^2 - \sum_{j=1}^{J} a_j K_\sigma(y+\lambda_j)\varphi(y) \right) \, dy \right| \\
&\leq \int_{-\infty}^{\infty} \left| \varphi(y)(\psi_n(y)\varphi(y) - \sum_{j=1}^{J} a_j K_\sigma(y+\lambda_j)) \right| \, dy \\
&\leq M \int_{-\infty}^{\infty} \left| \psi_n(y)\varphi(y) - \sum_{j=1}^{J} a_j K_\sigma(y+\lambda_j) \right| \, dy \\
&\leq M\epsilon.
\end{aligned}$$

But

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\psi_n(y)\varphi(y)^2 - \sum_{j=1}^{J} a_j K_\sigma(y+\lambda_j)\varphi(y)) \, dy \\
&= \int_{-\infty}^{\infty} \psi_n(y)\varphi(y)^2 \, dy - a_j \sum_{j=1}^{J} \int_{-\infty}^{\infty} K_\sigma(y+\lambda_j)\varphi(y) \, dy \\
&= \int_{-\infty}^{\infty} \psi_n(y)\varphi(y)^2 \, dy.
\end{aligned}$$

Therefore for all $\epsilon > 0$ and all $n \in \mathbb{N}$ we have

$$\left| \int_{-\infty}^{\infty} \psi_n(y)\varphi(y)^2 \, dy \right| < M\epsilon,$$

which implies $\int_{-\infty}^{\infty} \psi_n(y)\varphi(y)^2 \, dy = 0$. Since $\varphi(y)^2 \geq 0$ for all y we have $\varphi(y) = 0$ a.e on $[-n, n]$. Because this is true for all n we have $\varphi(y) = 0$ a.e. on $\mathbb{R}$. Applying this to the real and imaginary parts of φ gives $\varphi(y) = 0$ almost everywhere. I gratefully acknowledge Emilio Negrin for observing that the existing proof of Step (3) needed to be replaced and producing the new proof together with the author.

• page 142 line 2: replace $\sigma > \frac{1}{2}$ by $0 < \sigma < 1$ with $\sigma \neq \frac{1}{2}$.

• Emilio Negrin also provided a much simpler proof for Step (4) page 142: Assume $\xi(\rho) = \xi(\sigma + i\gamma) = 0$, for some $\sigma > 1/2$ and $\gamma \in \mathbb{R}$. Then, referring back to Step (1), the Fourier transform (at this point γ) of the kernel K_σ is zero. Thus, making the change of variables $u = x - y$ for $x \in \mathbb{R}$ one has for each x

$$\int_{-\infty}^{\infty} K_\sigma(x-y)e^{i(x-y)\gamma}dy = 0,$$

so

$$\int_{-\infty}^{\infty} K_\sigma(x-y)e^{-iy\gamma}dy = 0.$$

Therefore we have found a bounded measurable function which is not identically zero a.e. on $\mathbb{R}$, namely $\varphi(y) = e^{-iy\gamma}$ satisfying the equation

$$\int_{-\infty}^{\infty} K_\sigma(x-y)\varphi(y)dy = 0$$

for all $x \in \mathbb{R}$.

• page 151 Figure 9.1 and the corresponding line on page xiv: replace the age range (1907-2007) by (1917-2007).

• proposed new chapter: Ken Ono, Michael Griffin, Larry Rolen and Don Zagier have shown that all but a finite number of the Jensen polynomials for the Riemann Xi function are hyperbolic. A result of Polya is that RH is equivalent to all of these polynomials being hyperbolic. A report on this work, including the details of the Polya equivalence, would be included in a second edition.

See *Jensen polynomials for the Riemann zeta function and other sequences*, by Michael Griffin, Ken Ono, Larry Rolen, and Don Zagier Proceedings of the National Academy of Sciences (PNAS) **116** (2019), 11103–11110.

See also the commentary by Enrico Bombieri *New progress on the zeta function: from old conjectures to a major breakthrough*, PNAS **116** (2019), 11085–11086.